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Yang–Lee edge singularity in the one-dimensional long-range Ising model

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Abstract. The finite-range scaling method has been used to study the Yang–Lee edge singularity problem for the one-dimensional ferromagnetic Ising model with long-range interactions decaying with distance r, as $r^{-d-\sigma}$. The phase diagram and the critical exponent ν have been calculated for different values of σ by using finite ranges $L \leqslant 9$. The critical value σ_c dividing the long-range from the short-range critical regime has been discussed.

1. Introduction

In this paper, we study a one-dimensional $S=1/2$ ferromagnetic Ising system, with long-range interactions decaying with distance r as $r^{-(d+\sigma)}$, $\sigma>0$, placed in a pure imaginary magnetic field $\mathrm{i}h''$, described by the Hamiltonian

$$H = -\sum_{\substack{i,j=1 \\ i\neq j}}^{N} J_{ij}S_iS_j - \mathrm{i}h'' \sum_{i=1}^{N} S_i \tag{1}$$

where

$$J_{ij} = J_0/|i-j|^{1+\sigma} \qquad J_0>0.$$

The presence of a purely imaginary symmetry breaking field produces a phase transition of the second order, which belongs to a different universality class from the zero-field one. It arises in the study of density of zeros of the partition function in a complex field plain (Yang and Lee 1952) and corresponds to the Yang–Lee edge singularity (Fisher 1978). While the $h=0$ transition ($h=h''/k_BT$) is due to the cumulation of those zeros near the real axis in the complex field plane, the Yang–Lee edge singularity is produced by their cumulation around some particular point on the imaginary axis $h_0(T)$, when the temperature T is above the zero-field critical temperature. Although a non-Hermitian problem, it appears to be in the same universality class as a number of more 'realistic' problems, such as branching polymers in $d+2$ dimensions (Parisi and Sourlas 1981) or electronic localization in a random potential also in $d+2$ dimensions (Lubensky and McKane 1981).

The Yang–Lee edge singularity, as well as the density of zeros itself, has been extensively studied for different short-range (SR) interaction models (see, for example, a review of Kurtze and Fisher (1979) which gives a long list of references). Various methods were applied, from high-temperature expansions (Kortman and Griffiths

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1971), through different renormalization group (RG) techniques such as the ε-expansion (Fisher 1978), block renormalization (Uzelac *et al* 1979), and finite-size scaling (Uzelac and Jullien 1981) to scarce exact results for the Ising model in $d=1$ (Lee and Yang 1952), $d=2$ (Dhar 1983, Cardy 1982, 1985)) and $d=\infty$ (Baker and Moussa 1978).

Within the long-range (LR) interaction models this problem has been less explored, although the Yang–Lee theorem and its consequences still hold there.

A number of results can, however, be obtained without much effort by following the ε-expansion approach for d-dimensional n-vector model in the k-space performed by Fisher (1978), by adding a LR term proportional to k^σ to the corresponding Φ^3 Hamiltonian, so that it becomes

$$-H/k_BT=h\Phi_0-\tfrac{1}{2}\int \mathrm{d}^dk(ek^2+fk^\sigma)\cdot|\Phi_k|^2-w\int \mathrm{d}^dk\,\mathrm{d}^dk'\,\Phi_k\Phi_{k'}\Phi_{-k-k'}. \tag{2}$$

The RG transformations in the zeroth order approximation then give the mean field (MF) borderline $d^*=3\sigma$ and a MF critical exponent $\eta=2-\sigma$. By deriving other MF exponents e.g. $\beta=\frac{1}{2}$, $\nu=1/(2\sigma)$, one can check that the borderline agrees also with hyperscaling. Furthermore, the exponents in the non-trivial region can also be derived without going through the extensive ε-expansion calculations. It is sufficient to establish the fact that, as for the ordinary LR transitions, no new terms proportional to k^σ are generated by the ε-expansion, so that the exponent η remains equal to its MF value $\eta=2-\sigma$ to all orders of ε. Then, since the Yang–Lee transition has only one independent critical exponent (critical exponent $\Delta=1$), all other exponents can be derived, e.g. $\nu=2/(d+\sigma)$.

A non-trivial question that arises involves the position of the second borderline, separating the LR from SR regime. For the usual Φ^4 model this borderline $\sigma_c(d)$ is determined (Sak 1973) by the condition $2-\sigma_c=\eta_{SR}$, i.e., the LR and SR values of the exponent η match, and since in this case $\eta_{SR}>0$, σ_C never exceeds two. In the present transition, however, a similar argument would result in values of $\sigma_c>2$, which in the context of Hamiltonian (2) becomes meaningless, the small-k expansion ceasing to be valid. Similar difficulty arises in other problems involving a cubic term, e.g. those defined for disorder or the spin glass (Priest and Lubensky 1976; Chang and Sak 1984). Priest and Lubensky argue that the above mentioned criterion should be valid only for positive η_{SR}, while for $\eta_{SR}<0$ it only ensures the LR critical behaviour whenever $\sigma<2$. The calculations of Chang and Sak explicitly show that the exchange of stabilities between the LR and SR fixed points still formally occurs at $\sigma_c=2-\eta_{SR}$, even when $\eta_{SR}<0$. However, they consider this result as unphysical, since it leads to the already mentioned failure of the small-k expansion, and they assume the LR/SR crossover at $\sigma=2$ with discontinuous change of critical exponents. We prefer to consider this result as indecisive. One way to avoid this difficulty is to try to solve the problem directly in the real space.

Recently, we developed a real-space RG procedure, the finite-range scaling (FRS), suitable for treating models with LR interactions in one dimension. We apply it to the 1-D Yang–Lee edge singularity problem defined by (1). Varying σ will permit us to verify the conjectured values for the critical exponents as well as to discuss the LR/SR borderline problem within a new context.

The plan of the paper is as follows: the next section contains a short formulation of FRS method and the definition of the transfer matrix used, paying particular attention to the convergence of the FRS-calculated quantities; in the third section the numerical results for the phase diagram and the critical exponents are presented, the LR/SR crossover is discussed; the conclusion is summarized in section four.

2. Method and calculations

We apply the FRS method recently formulated for the 1-D Ising model (Uzelac and Glumac 1988) and explained in detail in our recent paper (Glumac and Uzelac 1989). Let us sketch its main idea and basic relations.

It is assumed that a physical quantity which, for small $t=(T-T_c)/T_c$, diverges in the infinite-range system as $C(t)=C_0t^{-\rho}$, can be written in homogeneous form

$$C_L(t)=L^{\rho/\nu}Y(L^{1/\nu}\cdot t, L^{y_3}\cdot u) \tag{3}$$

when the interaction range is truncated to the Lth neighbour. T_c and ρ, ν denote the critical temperature and the exponents of the infinite-range system respectively, while the parameter u represents the leading correction to scaling, with the corresponding irrelevant exponent y_3.

In particular, applied to the correlation length ξ, this assumption gives

$$\xi_L=LY_\xi(L^{1/\nu}\cdot t, L^{y_3}\cdot u). \tag{4}$$

The transition temperature is then numerically determined by the fixed point equation

$$\xi_L(t_L)/L=\xi_{L-1}(t_L)/(L-1) \qquad t_L=(T_L-T_c)/T_c \tag{5}$$

while the correlation length critical exponent ν is given by

$$\nu_L^{-1}=\ln(\xi'_L/\xi'_{L'})/\ln(L/L')-1 \tag{6}$$

where

$$\xi'=\mathrm{d}\xi/\mathrm{d}T \qquad \text{and} \qquad L'=L-1.$$

The finite-range quantities required are calculated by the transfer-matrix formalism used in the $h=0$ case (Glumac and Uzelac 1989). The transfer matrix $\mathfrak{T}$ is defined as

$$\mathfrak{T}_{j,j+1}=\exp\{-[H_{j,j+1}+(H_j+H_{j+1})/2]/T\} \tag{7}$$

where we choose the system of units where $J_0/k_B=1$, and

$$H_{j,j+1}=-\sum_{n=0}^{L-1}J_{L-n}\sum_{i=1}^{L-n}\alpha_j(i)\alpha_{j+1}(i+n) \tag{8}$$

$$H_j=-\sum_{n=1}^{L-1}J_n\sum_{i=1}^{L-n}\alpha_j(i)\alpha_j(i+n)+ih\sum_{i=1}^{L}\alpha_j(i) \tag{9}$$

where we have introduced the L-component variable α defined through $\alpha_j(i)=S_{(j-1)L+i}$ (see figure 1).

Note that the addition of the external field h breaks the symmetry of the Hamiltonian under the change $S_i\to -S_i$, so that the quasidiagonalization into two 2^{L-1}-th order matrices is not possible (unlike the $h=0$ case).

The correlation length for the range L is given by

$$\xi_L=L/\ln(\lambda_{1,L}/\lambda_{2,L}) \tag{10}$$

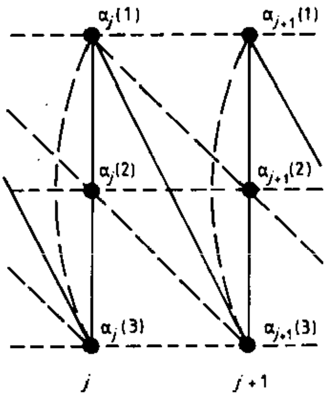

Figure 1. A part of the chain drawn in zig-zag form suggesting the use of transfer-matrix for $L=3$. Interactions of ranges $L=1$, 2, 3 are represented by full, long-broken and short-broken lines respectively.

where $\lambda_{1,L}>\lambda_{2,L}$ are the two eigenvalues of $\mathfrak{T}$ with the largest real parts. It is important to notice that, in the region of temperatures considered (near T_L), both of those eigenvalues are real. By lowering the temperature, another characteristic temperature (T_{cL}) is reached. It is the critical temperature of the second-order phase transition, which for the Yang–Lee singularity also exists in the finite-range system. At this point the eigenvalues merge and below it they become complex conjugate. Similar behaviour is encountered for the Yang–Lee transition in quantum 1-D Ising model (Uzelac and Jullien 1981).

The second quantity considered is the free energy density f_L, given by

$$f_L=-k_BTL^{-1}\ln(\lambda_{1,L})=f_0+f_{L\mathrm{Sing}} \tag{11}$$

where f_0 denotes the non-singular part. Its singular part f_{Sing} is also expected to have a homogeneous form (3)

$$f_{L\mathrm{Sing}}=L^{-d_f}\,Y_F(L^{1/\nu}\cdot t,\,L^{y_3}\cdot u) \tag{12}$$

where d_f should be equal to the dimension of space d.

The quantities obtained as T_L, ν_L or ξ_L/L are L-dependent due to different corrections, which should be examined more closely.

A peculiarity of the present problem in comparison with the $h=0$ one is the previously mentioned existence of two different second-order transitions: one in the $L=\infty$ limit (at T_c) and another for finite L (at T_{cL}). In order to study the resulting modifications of scaling, we proceed along the same lines as for such a case within the FSS (Ferdinand and Fisher 1969). As the result we obtain the following scaling form near T_{cL}

$$C_L(t)\propto L^{(\rho-\dot\rho)/\nu}\cdot t^{-\dot\rho} \tag{13}$$

where $\dot t=(T-T_{cL})/T_c$ and $\dot\rho$ is the finite-range exponent. Consequently, the L-dependence of the quantity $\xi_L(T_L)/L$ can adopt two types of behaviour:

$$\xi_L/L=a+bL^{-|y_3|} \tag{14}$$

for T_L approaching the T_c regime, and

$$\xi_L/L=cL^{\dot\nu|y_3|} \tag{15}$$

for T_L approaching the T_{cL} regime, where a, b, c are constants.

As regards the L-dependence of T_L and ν_L, the corrections are expected to be of the powerlaw type, if the corrections to scaling of the form (3) are assumed (Privman and Fisher 1983), i.e.

$$t_L = (T_L - T_c)/T_c = a_1 L^{-1/\nu + y_3} + \ldots, \tag{16}$$

$$\nu_L = \nu_e^{-1} + a_2 L^{1/\nu} \cdot t_L + a_3 L^{y_3} + \ldots = \nu_e^{-1} + a_4 L^{y_3} \tag{17}$$

where a_i and ν_e are constants.

Our numerical calculations have been performed up to the range $L=9$. Since the above described L-dependence of critical temperature and exponents is rather pronounced, a careful convergence study with appropriate extrapolation procedures is needed to draw the conclusion on the $L=\infty$ limit. In this paper we have used the following two extrapolation procedures:

(1) Vanden–Broeck and Schwartz (1979) extrapolation method with modifications due to Hamer and Barber (1981), described in our previous article (Glumac and Uzelac 1989). Since the VBS method requires great precision of the input data, it could be applied to T_L and $f_{L\mathrm{Sing}}$, but not to ν_L.

(2) Fitting the input data y_L to the powerlaw form

$$y_L = a + c_1 L^{-\alpha} \tag{18}$$

in the least-squares approximation (LSA).

Due to the small number of data in our case, both methods have to be used with great care. As a test we have compared the fit with the more extended form

$$y_L = a + c_1 L^{-\alpha} + c_2 L^{-\beta} \qquad L = 5\text{–}9 \tag{19}$$

by numerical solution of the 5×5 system of equations, where $a = \lim_{L\to\infty} y_L$, c_i are constants and α and β are convergence exponents.

3. Results

(a) Phase diagram

The phase diagram resulting from the scaling equation (5) is presented in figure 2 for several values of σ. The parameter space investigated is reduced to the region $0<h<\pi/2$ due to the invariance properties of the partition function

$$Z(h) = Z(h + m\pi) \qquad m = 0, \pm1, \pm2, \ldots \tag{20}$$

$$Z(h) = Z(\pi - h) \tag{21}$$

which follow from the periodicity in h and from the $S_i \to -S_i$ symmetry respectively.

K_{ce} denotes the inverse critical temperature, obtained by $L\to\infty$ extrapolation of L-dependent values $K_L = 1/T_L$ and defined by equation (5). The short-range result, given by the exact expression $K_c = -\ln(\sin h)/2$ is also drawn for comparison (the broken line).

We include on the abscissa our previous results for $h=0$ (Glumac and Uzelac 1989). It is interesting to notice the small humps near $h=0$ in the small σ region. We have checked that they are not caused by the difference in maximum range used in the two calculations (for $h=0$ the range was $L\leq 10$). Rather, they point out the

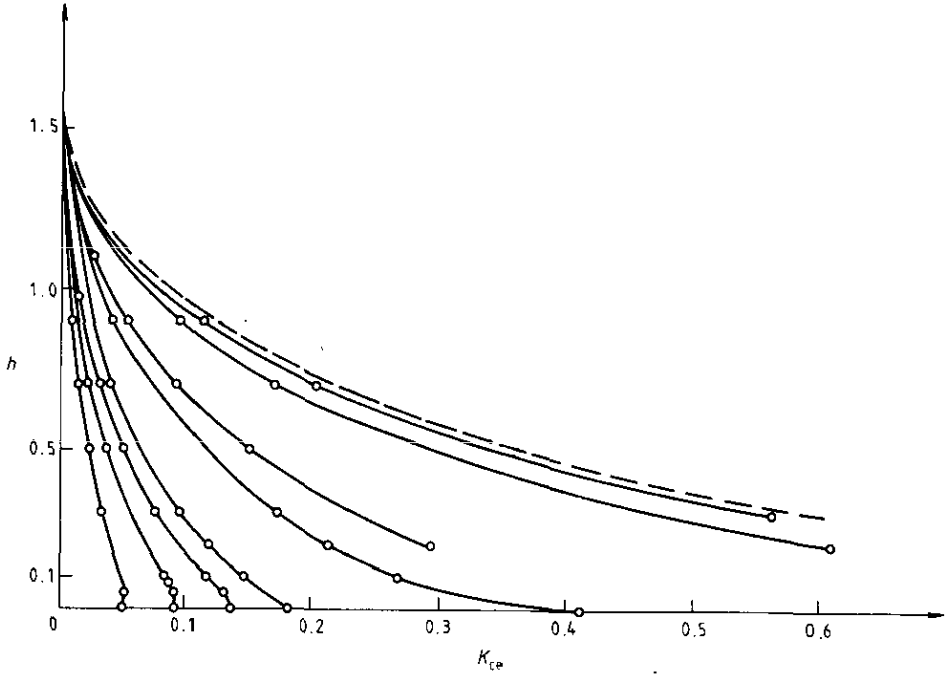

Figure 2. Phase diagram for LR interaction with following σ values: 0.1, 0.2, 0.3, 0.4, 0.8, 1.1, 2.5, 4.0 going from left to right. The exact SR result is traced by the broken curve.

difference between the convergence laws, belonging to two different phase transitions, which becomes more perceptible in the region of small σ, where this convergence is the weakest in both cases.

The L-dependence of K_L is illustrated in figure 3 for different σ's in the range 0.1–4, with a common fixed value of h.

(b) Critical behaviour

Before approaching the calculation of critical behaviour, let us examine the validity of the initial scaling hypothesis (4) for the present problem. It requires that the quantity ξ_L/L remains constant as a function of $1/L$, while the deviation from such behaviour indicates the importance of corrections to scaling. In figure 4 we present our numerical results for ξ_L/L as a function of $1/L$ for different σ's varying in the wide range 0.3–4, while L is taken from 4–8. There are two sets of results, taken at K_{ce} and K_L respectively, both for the same fixed $h = 0.7$. One can observe a good scaling behaviour for σ up to approximately 3, with no particular difference between the mean-field ($\sigma < \frac{1}{3}$) and the rest of this region. This should be expected, since, as we already pointed out in the $h = 0$ case, contrary to FSS (Brézin 1982), the FRS should remain valid in the MF region. The reason is that the basic scaling parameter is the range of scaling, which in the MF region remains relevant. The corrections to constant behaviour for $\sigma < 2.8$ have the regular form proportional to L^{-x}, decreasing with L. For $\sigma \geqslant 2.8$ the corrections start to increase with L, indicating the changing of the regime which will be discussed at the end of this section.

As mentioned earlier, it is sufficient to know one critical exponent. We calculate the exponent ν using (6). The important difference from our previous applications of FRS is the presence of an additional parameter h. Although the values of the critical

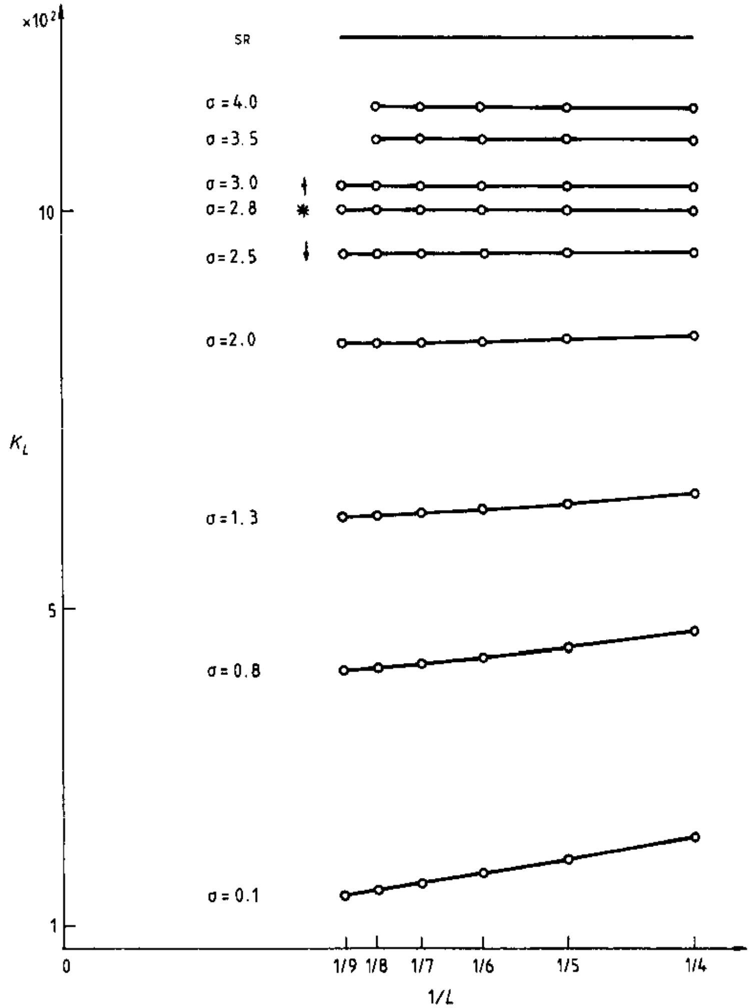

Figure 3. Dependence of FRS transition-temperature K_L on $1/L$ for different values of σ and fixed $h = 0.9$. Circles represent data for $L = 4$–9. Arrows pointing up and down indicate monotonically increasing and decreasing sequences of data, while * denotes the non-monotonical sequence. The full line denotes the SR transition-temperature at the same h.

exponents do not depend on the position on the line $T_c(h)$, our results turn out to vary considerably with it, since the secondary effects which influence the convergence of results are rather susceptible to the choice of h. Looking for the criterion to select our results, we choose to take those values of h for which the scaling relations are obeyed best. For this purpose, we calculate another quantity, the free energy density and search for the h for which the corresponding critical exponent d_f (c.f. (12)) is closest to unity, its exact value. The value of h obtained in this way is denoted by h^*. Table 1 presents, for different σ's, the values of h^* followed by the corresponding d_f, K_{ce} and the related convergence exponent x_K. The error bars for K_{ce} are taken as the difference between the two used extrapolation procedures.

The critical exponent ν has then been calculated from (6), by taking $h = h^*$. Since for $\sigma \leqslant 2.8$ the corrections to scaling are smaller for K_{ce} (c.f. figure 4), we have taken

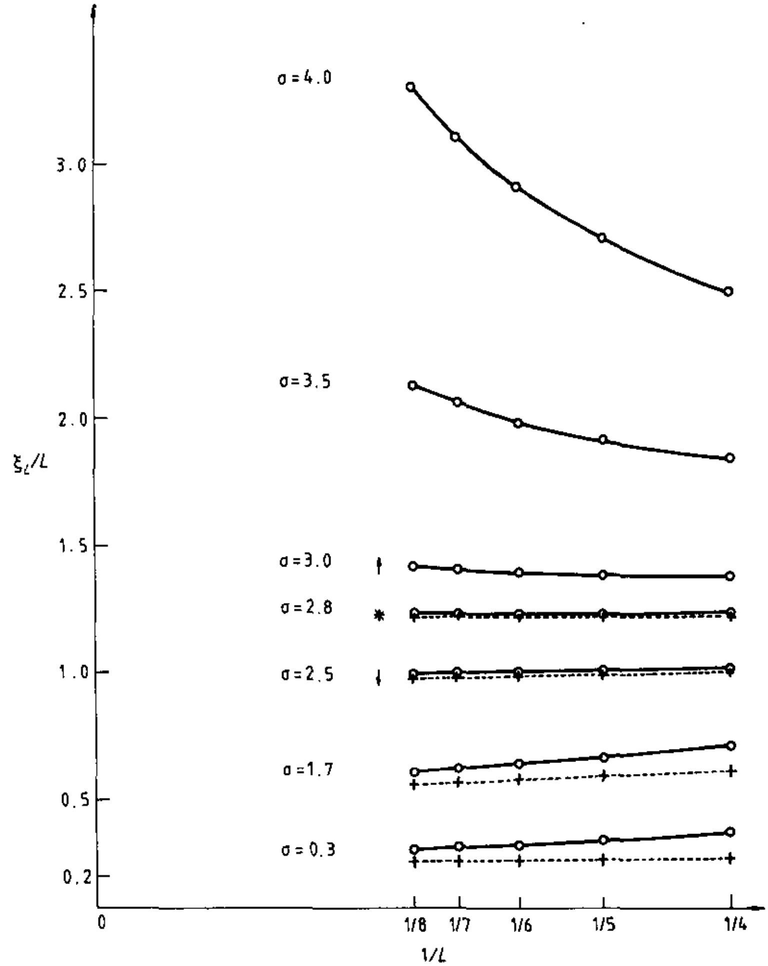

Figure 4. Results for ξ_L/L as a function of $1/L$ for different σ and fixed $h=0.7$. Data calculated at K_L and K_{ce} are represented by open circles and crosses respectively. Arrows pointing up and down indicate monotonically increasing and decreasing sequences of data, while * denotes the non-monotonical sequence.

$K_c = K_{ce}$ for $\sigma \leqslant 2.8$, while $K_c = K_L$ for $\sigma \geqslant 3$. Results are presented in table 1. Values of ν_e are obtained as $L \to \infty$, extrapolations from ν_L are calculated for sizes $L=7$–9. With the exception of the MF region, the comparison with conjectured values shows the agreement to within a few percent. A larger error is to be expected in the MF region since it involves small values of σ. As observed already for $h=0$ (Glumac and Uzelac 1989), the effect of truncation is more pronounced there which slows down the convergence of results.

In table 2 we compare the convergence exponents x_ν and x_ξ corresponding to ν_L and ξ_L/L respectively. According to (17) and (14) they should both be equal and related to the leading irrelevant critical exponent y_3. Indeed, we notice that the values for x_ν and x_ξ are quite close to each other. Unfortunately, conjecture giving the exponent ν does not provide any estimate for y_3, so we cannot make any comparison. Also, we point out that the results given at table 2 should be considered as a rough approximation, due to limitations of our convergence analysis mentioned earlier.

Table 1. The error bars are estimated to be lower than last cited digits.

σ	h^*	d_f	K_{ce}	x_K	ν_e	ν_{exact}
0.1	0.3	0.93	0.035	1.05	4	5.00
0.2	0.08	0.98	0.087	0.99	1.8	2.50
0.3	0.1	0.97	0.116	0.98	1.4	1.67
0.4	0.1	0.97	0.147	0.95	1.3	1.43
0.8	0.25	0.94	0.192	1.04	1.02	1.11
1.3	0.4	0.95	0.218	1.05	0.88	0.87
1.5	0.7	0.97	0.120	1.13	0.86	0.80
1.7	0.9	0.99	0.0744	1.34	0.70	0.74
2.0	0.7	0.92	0.1483	1.65	0.68	0.67
2.2	0.8	0.96	0.1196	1.61	0.63	0.63
2.5	0.9	1.00	0.0947	1.10	0.60	0.57
2.8	1.0	1.02	0.0709	1.15	0.56	0.53
3.0	0.9	0.94	0.1032	0.82	0.51	0.50
3.2	0.7	1.00	0.1901	1.01	0.51	0.50
3.5	0.7	0.94	0.1962	1.66	0.50	0.50
4.0	0.3	0.95	0.5637	1.58	0.50	0.50

Table 2. Convergence exponents x_ξ and x_ν defined by relations (9) and (12) respectively, as functions of σ for the same choice of h^*. The error bars are estimated to be lower than last cited digits.

σ	h^*	x_ξ	x_ν
0.1	0.3	0.98	1.71
0.2	0.08	1.04	1.27
0.3	0.1	0.94	1.11
0.4	0.1	0.95	1.05
0.8	0.25	0.81	0.31
1.3	0.4	0.88	0.94
1.5	0.7	0.87	0.95
1.7	0.9	0.83	1.22
2.0	0.7	0.94	1.04
2.2	0.8	1.00	1.12
2.5	0.9	0.45	7.43
2.8	1.0	0.39	1.64
3.0	0.9	—	0.47
3.2	0.7	—	0.72
3.5	0.7	—	0.81
4.0	0.3	—	1.62

Turning back to the question of σ_c, the boundary between the LR and SR regimes, let us consider figure 5 which represents ν^{-1} as a function of σ. Our numerical results are compared to the MF LR value $\nu^{-1}=2\sigma$, the conjectured value $\nu^{-1}=(d+\sigma)/2$ in the non-trivial LR region and the SR value $\nu_{SR}^{-1}=2$. The figure does not show the jump at $\sigma=2$, but rather a continuous crossover at $\sigma=3$ with the deviation from conjectured values not exceeding 6%.

The two figures discussed earlier also indicate the change near $\sigma_c=3$. In figure 4, representing ξ_L/L as a function of $1/L$ and σ, two different scaling regimes (equations (14) and (15)) have been observed, separated at $\sigma=2.8$. The same point is characteristic

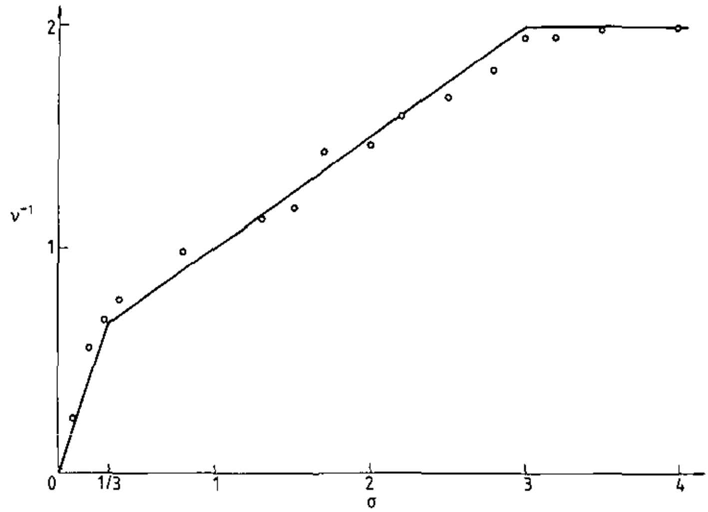

Figure 5. Extrapolated values for ν^{-1} (open circles) compared with the exact or conjectured values (full line).

for the change of behaviour of K_L as a function of $1/L$ (figure 3), which is manifested through the change from monotonously decreasing to monotonously increasing behaviour. The analogous change in the behaviour of critical temperature has already been observed in the case $h=0$ (Glumac and Uzelac 1989), where it corresponds also to the LR/SR boundary, which in that case occurs at $\sigma=1$.

Although it cannot be ruled out that the continuous change of exponents (instead of a jump) is a consequence of our FRS procedure, we could not find arguments that would make such an explanation preferable. On the other hand, the results listed above strongly support the continuous change from the LR to the SR regime at $\sigma=3$.

4. Conclusion

We have shown that the recently developed FRS method can be applied to the Yang–Lee edge singularity in the 1-D LR Ising model. After careful handling of the additional parameter, and using appropriate extrapolation procedures, rather precise results for the phase diagram and the critical exponent ν were obtained. The exponent ν agrees with the value following from scaling relations with $\Delta=1$ and $\eta=2-\sigma$.

Since the method applies in direct space, the question of localization of LR/SR crossover value of σ could be considered out of context of the small-k expansion. Although an artefact of the method cannot be excluded, the obtained results support $\sigma_c=3$ with the continuous change of critical exponents.

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References

Baker G A and Moussa P 1978 *J. Appl. Phys.* **49** 1360
Brézin E 1982 *J. Physique* **43** 15
Cardy J L 1982 *J. Phys. A: Math. Gen.* **15** L593
—— 1985 *Phys. Rev. Lett.* **54** 1354
Chang M C and Sak J 1984 *Phys. Rev.* B **29** 2652
Dhar D 1983 *Phys. Rev. Lett.* **51** 853
Ferdinand A E and Fisher M E 1969 *Phys. Rev.* **185** 832
Fisher M E 1978 *Phys. Rev. Lett.* **40** 1610
Glumac Z and Uzelac K 1989 *J. Phys. A: Math. Gen.* **22** 4439
Hamer C J and Barber M N 1981 *J. Phys. A: Math. Gen.* **14** 2009
Kortman J P and Griffiths R B 1971 *Phys. Rev. Lett.* **27** 1439
Kurtze D A and Fisher M E 1979 *Phys. Rev.* B **20** 2785
Lee T D and Yang C N 1952 *Phys. Rev.* **87** 410
Lubensky T C and McKane A F 1981 *J. Physique Lett.* **42** L331
Parisi G and Sourlas N 1981 *Phys. Rev. Lett.* **46** 871
Priest R G and Lubensky T C 1976 *Phys. Rev.* B **13** 4159
Privman V and Fisher M E 1983 *J. Phys. A: Math. Gen.* **16** L295
Sak J 1973 *Phys. Rev.* B **8** 281
Uzelac K and Glumac Z 1988 *J. Phys. A: Math. Gen.* **21** L421
Uzelac K and Jullien R 1981 *J. Phys. A: Math. Gen.* **14** L151
Uzelac K, Pfeuty P and Jullien R 1979 *Phys. Rev. Lett.* **43** 805
Yang C N and Lee T D 1952 *Phys. Rev.* **87** 404
Vanden Broeck J M and Schwartz L W 1979 *SIAM J. Math. Anal.* **10** 658